ON THE STABILITY OF A HIGHLY CONDUCTING PLASMA STRING IN A MAGNETIC FIELD

(OB USTOICHIVOSTI KHOROSHO PROVODIASHCHEGO PLAZMENNOGO SHNURA V MAGNITNOM POLE)

PMM Vol.27, No.1, 1963, pp. 46-54

Iu. V. VANDARUKOV
(Leningrad)

(Received July 7, 1962)

The investigation of the stability problem of a plasma string supported by a magnetic field when finite conductivity is taken into account has been considered in [1-3]. In the case of an incompressible string of homogeneous conductivity, the regions of instability were established for certain types of oscillations [1,2].

The stability of a pinch with surface currents in a magnetic layer of large, but finite plasma conductivity has been studied by Jukes [3]. In this work all the calculations were done under the assumption that the radial component of a displacement ξ_r from the position of equilibrium within the confines of the magnetic layer remains constant. Accordingly, a solution is found of an incomplete system of equations of magnetohydrodynamics, However, whereas in the case of an ideally conducting medium the assumption of the stability of ξ_r is a consequence of the resulting equations [4], this assumption turns out to be incorrect in the general case, as will be seen from the solutions given below. Therefore, the results of the work by Jukes should be considered as incorrect.

Let us investigate the stability of a plasma string in the form of a hollow cylinder under the assumption that the conductivity σ is a significant parameter of the problem. No other limitations on the form of function $\sigma(r)$ are imposed.

The formulation of the stability problem for a pipe-like string of finite conductivity in the case of an incompressible medium has been treated in [4]; we shall derive the equations with consideration of the compressibility of the medium (Section 1); Sections 2 to 4 deal with the stability of an incompressible hollow cylinder; Sections 5 and 6 are devoted to a study of a solid incompressible cylinder and of a pipe-like string with consideration of the compressibility of the medium.

The solutions obtained pertain to unstable oscillations with purely exponential dependence on time. These instabilities are derived from discontinuities of the spatial steady-state distribution of the current density.

1. Basic equations. Consider a steady state plasma layer occypying the space between two coaxial cylinders with radii r_1 and r_2. The pressure, density and conductivity of the layer are $p(r)$, $\rho(r)$ and $\sigma(r)$. Let us represent the azimuthal and the axial components of the magnetic in the form

$$H_\varphi = H_0 \frac{rg(r)}{r_0}, \qquad H_z = H_0 h(r), \qquad g(r_0) = 1, \qquad H_0 = (H_\varphi)_{r=r_0} \neq 0 \quad (1.1)$$

where r_0 is some intermediate radius of the layer.

Let it be assumed that the plasma layer is surrounded by ideally conducting cylinders. The inner cylinder has a radius $\alpha_1 r_1$ and the outer a radius $\alpha_2 r_2$. In the non-conducting layer $\alpha_1 r_1 \leqslant r \leqslant r_1$ the pressure and density are constant and equal to p_1 and $\chi_1 \rho_1$, respectively. The corresponding quantities for the layer $r_2 \leqslant r \leqslant \alpha_2 r_2$ have the subscript 2. A small disturbance is imposed upon the given equilibrium distribution, defined by the function exp $i(\omega t + m\varphi + kz)$. As in [4], assume that $m \geqslant 0$ and $k \geqslant 0$. As the resulting equations of magnetohydrodynamics are linearized they become*

$$\begin{gathered}
\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \frac{1}{4\pi} \mathbf{H} \times \operatorname{rot} \mathbf{H}, \qquad \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 \\
\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot} (\mathbf{v} \times \mathbf{H}) - \frac{c^2}{4\pi} \operatorname{rot} \left(\frac{1}{\sigma} \operatorname{rot} \mathbf{H} \right), \qquad \operatorname{div} \mathbf{H} = 0 \\
\frac{d}{dt} (p\rho^{-\gamma}) = 0, \qquad \gamma = \text{const}
\end{gathered} \qquad (1.2)$$

we obtain

$$\frac{\Omega^2 H_0}{r_0} \boldsymbol{\xi} = -\nabla Q + is\mathbf{H}^* - 2\mathbf{i}_r g H_\varphi^* + \{\mathbf{i}_\varphi (rg' + 2g) + \mathbf{i}_z r_0 h'\} H_r^* \qquad (1.3)$$

---

* Strictly speaking, instead of the isentropic equations of motion we should write the equations of energy transfer, in which the Joule losses and the heat conductivity are taken into account. However, in the case under consideration of a well conductive medium, the contribution of Joule losses leads to insignificant corrections. The isothermal condition may be obtained, if we take $\gamma = 1$ and if we assume that in the steady condition $(r)/\rho\ (r)$ = const.

$$\mathbf{H}^* = \frac{isH_0}{r_0}\boldsymbol{\xi} - (\mathbf{i}_\varphi rg' + \mathbf{i}_z r_0 h')\frac{H_0\xi_r}{r_0} + \frac{r_0^2}{\Omega q^2}\left\{\nabla^2\mathbf{H}^* + \frac{\sigma'}{\sigma}\mathbf{i}_r \times \operatorname{rot}\mathbf{H}^*\right\} - \mathbf{H}\operatorname{div}\boldsymbol{\xi} \quad (1.4)$$

$$p^* = -\gamma p \operatorname{div}\boldsymbol{\xi} - p'\xi_r, \qquad \rho^* = -\operatorname{div}\rho\boldsymbol{\xi} \quad (1.5)$$

$$Q = \frac{r_0}{H_0}(4\pi p^* + \mathbf{H}\cdot\mathbf{H}^*) \quad (1.6)$$

$$\boldsymbol{\xi} = \frac{\mathbf{v}^*}{i\omega}, \qquad \Omega = \frac{i\omega r_0\sqrt{4\pi\rho}}{|H_0|}, \qquad q^2 = \frac{r_0\sigma|H_0|}{c^2}\sqrt{\frac{4\pi}{\rho}}, \qquad s = mg + kr_0h$$

where $\mathbf{i}_r$ $\mathbf{i}_\varphi$ and $\mathbf{i}_z$ are unit vectors, the starred quantities are the excited quantities and the primes denote differentiation with respect to r.

Expressing $\boldsymbol{\xi}$ in terms of $\mathbf{H}^*$ and div $\boldsymbol{\xi}$ from (1.4) and including p^* by the use of (1.5), we convert the system (1.3) to (1.6) into the form

$$\nabla Q = \mathbf{i}_r P_1 + \mathbf{i}_\varphi i P_2 + \mathbf{i}_z \frac{iP_3}{kr} + \frac{i\Omega^2}{s}\mathbf{H}\operatorname{div}\boldsymbol{\xi} \quad (1.7)$$

$$Q = rgH_\varphi^* + r_0 hH_z^* - \frac{4\pi r_0}{H_0}\left\{\gamma p\operatorname{div}\boldsymbol{\xi} - \frac{ir_0p'}{sH_0}\left(H_r^* - \frac{r_0^2}{\Omega q^2}\nabla_r^2\mathbf{H}^*\right)\right\} \quad (1.8)$$

$$\operatorname{div}\mathbf{H}^* = 0 \quad (1.9)$$

$$P_1 = i\left(s + \frac{\Omega^2}{s}\right)H_r^* - 2gH_\varphi^* - \frac{ir_0^2\Omega}{q^2 s}\nabla_r^2\mathbf{H}^* \quad (1.10)$$

$$P_2 = \left(s + \frac{\Omega^2}{s}\right)H_\varphi^* - \left[\left(1 + \frac{\Omega^2}{s^2}\right)rg' + 2g\right]iH_r^* - \\ - \frac{r_0^2\Omega}{q^2 s}\left(\nabla_\varphi^2\mathbf{H}^* - \frac{\sigma'}{\sigma}\operatorname{rot}_z\mathbf{H}^* - \frac{irg'}{s}\nabla_r^2\mathbf{H}^*\right) \quad (1.11)$$

$$P_3 = \left(s + \frac{\Omega^2}{s}\right)krH_z^* - ikr_0rh'\left(1 + \frac{\Omega^2}{s^2}\right)H_r^* - \\ - \frac{krr_0^2\Omega}{q^2 s}\left(\nabla^2 H_z^* + \frac{\sigma'}{\sigma}\operatorname{rot}_\varphi\mathbf{H}^* - \frac{ir_0h'}{s}\nabla_r^2\mathbf{H}^*\right) \quad (1.12)$$

where $\nabla_\varphi^2\mathbf{H}^* = \mathbf{i}_\varphi \quad \nabla^2\mathbf{H}^*$ and so on; also

$$\frac{H_0}{r_0}\xi_r = -\frac{i}{s}\left(H_r^* - \frac{r_0^2}{\Omega q^2}\nabla_r^2\mathbf{H}^*\right) \quad (1.13)$$

To determine the boundary conditions the regions outside the conducting layer must be investigated. Calculations, analogous to those presented in [4], lead to the relations

$$\left\{mQ + \frac{r\chi\Omega^2 H_0\xi_r}{r_0 T^{(*)}} - \frac{isr}{T}H_r^*\right\}_{r=r_j} = 0 \quad (1.14)$$

$$\left\{H_\varphi^* - \frac{i}{T}H_r^* + (rg' + 2g)\frac{H_0\xi_r}{r_0}\right\}_{r=r_j} = 0 \quad (1.15)$$

$$\left\{H_z^* + h'H_0\xi_r - \frac{ikr}{mT} H_r^*\right\}_{r=r_j} = 0 \tag{1.16}$$

where

$$T^{(*)} = \frac{\varkappa r\,[I_m'(\varkappa r)\,K_m'(\varkappa \alpha r) - K_m'(\varkappa r)\,I_m'(\varkappa \alpha r)]}{m\,[I_m(\varkappa r)\,K_m'(\varkappa \alpha r) - K_m(\varkappa r)\,I_m'(\varkappa \alpha r)]}$$

$$\varkappa = \sqrt{k^2 - \omega^2\rho / \gamma p}, \qquad (\chi)_{r=r_j} = \chi_j \qquad (\alpha)_{r=r_j} = \alpha_j \qquad (j = 1, 2)$$

T is determined by the same equation as $T^{(*)}$ through substitution of K for κ.

The investigation of stability with respect to a given type of disturbances now consists in finding Ω which must satisfy the system (1.7) to (1.9) under the conditions (1.14) to (1.16). For unstable oscillations Re Ω > 0.

2. Asymptotic solutions in the case of an incompressible medium. In the case of incompressibility div ξ = 0, the quantity γ div ξ is finite and the equation (1.8) may be disregarded. A solution is desired of the system (1.7) and (1.9) for the case of large q, under the assumptions that the inside radius of the conducting layer is not small, that the function $s(r)$ is not near zero and that the equilibrium distributions are smooth (g' is of the order g/r, etc.).

Applying operator rot to equation (1.7), we obtain three equations, two of which are independent. As a result we arrive at the following system of three equations for the components of $\mathbf{H}^*$:

$$P_3 - \frac{k^2r^2}{m} P_2 = 0, \qquad \frac{\partial r P_2}{\partial r} - mP_1 = 0 \tag{2.1}$$

$$i\frac{\partial r H_r^*}{\partial r} = mH_\varphi^* + krH_z^* \tag{2.2}$$

The quantity $1/q^2$ in equation (2.1) appears as a small parameter for the higher derivative. Consider solutions, for which $\partial/\partial r \gg 1/r$. If only the major terms are retained, it can be shown that the components of $\mathbf{H}^*$ are proportional to exp λf, where

$$\lambda^2 f'^2 = \frac{q^2(s^2 + \Omega^2)}{\Omega r_0^2} \tag{2.3}$$

Assume, that derivative f' is of the order of $1/r_0$ and is not equal to zero. Then λ = const will be a significant parameter. It may be assumed that

$$\lambda = \left| q\sqrt{\frac{s^2+\Omega^2}{\Omega}} \right|_{r=r_0}$$

We shall look for solutions of (2.1) in the form

$$H_\varphi^* = \lambda f' [Y + \lambda^{-1} Y^{(1)} + O(\lambda^{-2})] e^{\lambda f}$$
$$H_z^* = \frac{m\lambda f'}{kr} [Z + \lambda^{-1} Z^{(1)} + O(\lambda^{-2})] e^{\lambda f} \qquad (2.4)$$

where Y, $Y^{(1)}$, Z, ... are functions of r, such that Y' is of the order of Y/r, etc.

From equations (2.2), (1.13) and (1.7) we find

$$H_r^* = -\frac{im}{r}\left[Y + Z + \frac{1}{\lambda}(Y^{(1)} + Z^{(1)}) - \frac{1}{\lambda f'}(Y' + Z') + O(\lambda^{-2})\right] e^{\lambda f} \qquad (2.5)$$

$$\frac{H_0 \xi_r}{r_0} = \frac{ms}{r\Omega^2} [Y + Z + \lambda^{-1} R^{(1)} + O(\lambda^{-2})] e^{\lambda f} \qquad (2.6)$$

$$Q = -2g [Y + O(\lambda^{-1})] e^{\lambda f} \qquad (2.7)$$

$$R^{(1)} = Y^{(1)} + Z^{(1)} + \frac{1}{f'}\left(1 + \frac{2\Omega^2}{s^2}\right)(Y' + Z') + \frac{1}{rf'}\left(1 + \frac{\Omega^2}{s^2}\right)\left[Y - Z + \frac{rf''}{f'}(Y + Z)\right]$$

Substituting (2.4) and (2.5) into the system (2.1) and equating the main terms to zero, we obtain (2.3). In the next approximation in terms of λ^{-1} we have

$$2Y' + \left(\frac{q'}{q} + \frac{3ss'}{s^2 + \Omega^2} + \frac{1}{r}\right) Y + \frac{2msg}{r(s^2 + \Omega^2)} Z = 0$$
$$2Z' + \left(\frac{q'}{q} + \frac{3ss'}{s^2 + \Omega^2} - \frac{1}{r}\right) Z - \frac{2k^2 rsg}{m(s^2 + \Omega^2)} Y = 0 \qquad (2.8)$$

Two linearly independent solutions of this system are obtained in the form

$$Y_1 = r_0 X \cos k\vartheta, \qquad Y_2 = -\frac{m}{k} X \sin k\vartheta, \qquad Z_1 = \frac{kr_0 r}{m} X \sin k\vartheta \qquad (2.9)$$

$$Z_2 = rX \cos k\vartheta \qquad X = \frac{C}{\sqrt{rq}} (s^2 + \Omega^2)^{-\frac{3}{4}}, \qquad \vartheta = \int_{r_1}^{r} \frac{sg\,dr}{s^2 + \Omega^2}, \qquad C = \text{const}$$

Continuing the separation expressions for $Y_n^{(1)}$ and $Z_n^{(1)}$, $n = 1, 2$ can also be established.

Since the sign of λ in equation (2.3) is arbitrary, four independent solutions of the system (2.1) to (2.2) may be derived. Assume

$$\lambda f = \frac{1}{r_0} \int_{r_1}^{r} q \sqrt{\frac{s^2 + \Omega^2}{\Omega}}\, dr, \qquad \left|\arg \sqrt{\frac{s^2 + \Omega^2}{\Omega}}\right| \leqslant \frac{\pi}{2} \qquad (2.10)$$

then two solutions are defined by the equations (2.4), (2.5) after substitution of (2.9) therein, and the other two will be obtained by changing the sign in front of λ. For example, the solutions for $\mathbf{H}_\varphi^*$ are $\pm\lambda f'(Y_n + \ldots)e^{\pm\lambda f}$, $n = 1, 2$.

The derived equations are valid under the conditions

$$\lambda \gg 1, \qquad \lambda \gg (m^2 + k^2r^2), \qquad |\lambda r f'| \gg 1, \quad (\lambda r_1/r_2) \gg 1$$
$$\lambda \gg |r\sigma'/\sigma|, \qquad \lambda \gg |rg'/s|, \qquad \lambda \gg |rh'/s|$$

Two last conditions exclude from consideration the case $s = 0$.

3. Dispersion relation. A complete system of solutions of the equations (2.1), (2.2) consists of six particular solutions. The last two solutions may be expected in the form of an expansion in terms of $1/\lambda^2$, assuming that the first terms of the series will be solutions to the approximation of ideal conductivity. Also the parameter λ must satisfy the condition $r_1{}^2\lambda^2 \gg r_2{}^2$. We assume such solutions to be known and denote the left-hand sides of the equations (1.14) to (1.16), after substitution of the nth solution ($n = 1, 2$), by U_{nj}, V_{nj}, W_{nj}, where $U_{nj} = U_n(s_j)$ etc., respectively.

Upon substitution of the complete solution into the conditions (1.14) to (1.16), we obtain

$$\begin{aligned}
A_1U_{1j} + A_2U_{2j} + (B_1D_{1j} + B_2D_{2j})\,e^{\lambda f_j} + (B_3D_{1j}{}^- + B_4D_{2j}{}^-)\,e^{-\lambda f_j} = 0 \\
A_1V_{1j} + A_2V_{2j} + (B_1E_{1j} + B_2E_{2j})\,e^{\lambda f_j} + (B_3E_{1j}{}^- + B_3E_{2j}{}^-)\,e^{-\lambda f_j} = 0 \qquad (3.1) \\
A_1W_{1j} + A_2W_{2j} + (B_1F_{1j} + B_2F_{2j})\,e^{\lambda f_j} + (B_3F_{1j}{}^- + B_4F_{2j}{}^-)\,e^{-\lambda f_j} = 0
\end{aligned}$$

where $j = 1, 2$

$$D_{nj} = -\,2mg_jY_{nj} - \frac{ms_j(1-\chi_j)}{T_j}\,(Y_{nj} + Z_{nj}) + O(\lambda^{-1})$$

$$E_{nj} = \lambda f_j'\left(Y_{nj} + \frac{1}{\lambda}\,Y_{nj}^{(1)}\right) + \left[\frac{ms_j}{r_j\Omega^2}(r_jg_j' + 2g_j) - \right. \qquad (3.2)$$
$$\left. - \frac{m}{r_jT_j}\right](Y_{nj} + Z_{nj} + \lambda^{-1}R_{nj}^{(1)}) + O(\lambda^{-2})$$

$$F_{nj} = \lambda f_j'\left(Z_{nj} + \frac{1}{\lambda}\,Z_{nj}^{(1)}\right) + \left(\frac{kr_0s_jh_j'}{\Omega^2} - \frac{k^2r_j}{mT_j}\right)\left(Y_{nj} + Z_{nj} + \frac{1}{\lambda}\,R_{nj}^{(1)}\right) + O(\lambda^{-2})$$

$$Y_{nj} = Y_n(r_j), \qquad h_j' = \left(\frac{dh}{dr}\right)_{r=r_j} \qquad (A_p,\ B_p = \text{const})$$

Equations for D^-, E^- and F^- are obtained from the equations for D, E and F by reversing the sign in front of λ. The factor in front of Ω^{-1}

has been computed with consideration of the correction terms, because the order of Ω is not known beforehand (it will be shown below that for $s \sim 1$ the parameters λ and Ω^{-2} may be of the same order).

By equating the determinant of the system (3.1) to zero, a solution for Ω is obtained. If the terms with the factor $\exp[-\lambda(f_2 - f_1)]$ are disregarded, we have

$$\begin{vmatrix} U_{11} & U_{21} & 0 & 0 & D_{11}^- & D_{21}^- \\ U_{12} & U_{22} & D_{12} & D_{22} & 0 & 0 \\ V_{11} & V_{21} & 0 & 0 & E_{11}^- & E_{21}^- \\ V_{12} & V_{22} & E_{12} & E_{22} & 0 & 0 \\ W_{11} & W_{21} & 0 & 0 & F_{11}^- & F_{21}^- \\ W_{12} & W_{22} & F_{12} & F_{22} & 0 & 0 \end{vmatrix} = 0$$

or in another form

$$\begin{aligned} &G_1G_2(U_{11}U_{22}-U_{12}U_{21})-G_1(D_{22}F_{12}-D_{12}F_{22})(U_{11}V_{22}-U_{21}V_{12})+ \\ &+G_1(D_{22}E_{12}-D_{12}E_{22})(U_{11}W_{22}-U_{21}W_{12})-G_2(D_{11}^-F_{21}^--D_{21}^-F_{11}^-)\times \\ &\times(U_{22}V_{11}-U_{12}V_{21})+G_2(D_{11}^-E_{21}^--D_{21}^-E_{11}^-)(U_{22}W_{11}-U_{12}W_{21})=0 \quad (3.3) \end{aligned}$$

where

$$G_1 = E_{11}^-F_{21}^- - E_{21}^-F_{11}^-, \qquad G_2 = E_{22}F_{12} - E_{12}F_{22}$$

and the neglected additives are λ^2 times smaller than the first term on the left-hand side of (3.3).

The expression for G_j may be written in the form

$$\begin{aligned} \frac{G_j}{N_j} &= \lambda f_j'\left[\lambda f_j' - (-1)^j\frac{m^2+k^2r_j^2}{mr_jT_j}\right] + \frac{s_j}{2r_j\Omega^2}\Big\{(rs'+2mg)\Big[(-1)^j 2\lambda f' + \\ &+\frac{q'}{q} - \frac{ss'}{s^2+\Omega^2}\Big] + mg' + \frac{2mg}{r} - kr_0h' + \frac{2k^2rsg}{s^2+\Omega^2}\Big(rg'+2g-\frac{mr_0h'}{kr}\Big)\Big\}_{r=r_j} \quad (3.4) \end{aligned}$$

In the equation (3.4)

$$N_j = (Y_{jj}+\lambda^{-1}Y_{jj}^{(1)})(Z_{lj}+\lambda^{-1}Z_{lj}^{(1)}) - (Y_{lj}+\lambda^{-1}Y_{lj}^{(1)})(Z_{jj}+\lambda^{-1}Z_{jj}^{(1)})$$
$$l = j + (-1)^{j-1}$$

It is seen easily that upon dividing by the factor N_1N_2, which differs from zero, the unknown functions $Y^{(1)}$ and $Z^{(1)}$ vanish from equation (3.3). The solution obtained represents the first two terms of the expansion into series in terms of λ^{-1} of an exact dispersion relation.

4. Investigation of the dispersion relation. In first approximation the roots of equation (3.3) are determined by equating the first term to zero. In addition to solutions similar to those which are obtained in the approximation of ideal conductivity, when

$$U_{11}U_{22} - U_{12}U_{21} = 0 \tag{4.1}$$

we obtain also solutions from $G_j = 0$ or

$$\lambda f_j' = (-1)^{j-1} \frac{s_j(r_j s_j' + 2mg_j)}{r_j \Omega^2} \qquad (j = 1, 2) \tag{4.2}$$

The complex roots Ω of the last expression, taking into account (2.10) correspond to attenuated oscillations. In case of unstable solutions the parameter Ω is real and positive. It is defined by the equations

$$q_1 = \frac{r_0 s_1 (r_1 s_1' + 2mg_1)}{r_1 \Omega \sqrt{\Omega (s_1^2 + \Omega^2)}}, \qquad s_1 (r_1' s_1' + 2mg_1) > 0 \tag{4.3}$$

$$q_2 = -\frac{r_0 s_2 (r_2 s_2' + 2mg_2)}{r_2 \Omega \sqrt{\Omega (s_2^2 + \Omega^2)}}, \qquad s_2 (r_2' s_2' + 2mg_2) < 0 \tag{4.4}$$

where

$$rs' + 2mg = \frac{r_0}{H_0}\left(\frac{m}{r}\frac{drH_\varphi}{dr} + kr\frac{dH_z}{dr}\right)$$

is a quantity related to the steady state current density $\mathbf{j}$. It is proportional to the scalar product of a gradient of some perturbed quantity (e.g. ∇H_φ^*) on $\mathbf{i}_r \times \mathbf{j}$.

Equations (4.3) and (4.4) define the frequencies of unstable oscillations with purely exponential dependence on time. For a given instant during the development of the instability

$$\tau = \frac{1}{i\omega} = \frac{r_0 \sqrt{4\pi\rho}}{\Omega |H_0|} \tag{4.5}$$

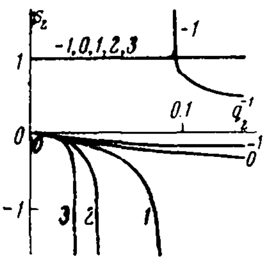
we may find from (4.3) and (4.4) the values of q, i.e. the conductivities, provided, of course, the conditions $\lambda f_j' \gg 1$ and those indicated in (4.3) and (4.4) are fulfilled.

For a more detailed study of the instabilities obtained it is necessary to carry out an analysis to the first and second approximations. We shall confine ourselves to the case of long wave disturbances ($k^2r^2 \ll 1$), when h' is of the order of k. If we neglect the quantities of the order k^2r^2, then in (3.3) we must assume that $Z_1 = 0$, $W_{nj} = 0$, $F_{11}^- = 0$ and $F_{12} = 0$. As a result equation (3.3) is greatly simplified. This approximation corresponds to neglecting the small term ikH_z^*, in the equation div $\mathbf{H}^* = 0$. In this case we can disregard the zth components of (1.3) and (1.4) together with the condition (1.16).

If we assume that $m = 1$, we can make use of the solution presented in [4, Section 3] for the calculation of the quantity of U_{nj} and V_{nj}. In the case of a solid string of homogeneous density for $\chi_2 = 0$, $T_2 = -1$, in the absence of currents in the region $r < r_1$, the dispersion relation (3.3) transforms into the form

$$\frac{G_1}{N_1}\Big\{(\Omega^2+2s^2-2sg)\Big[\lambda r f'\Big(\lambda r f' + \frac{srg'+2sg}{\Omega^2}+1\Big)+ \frac{srg'+2sg}{2\Omega^2}\Big(1+\frac{rq'}{q}-\frac{rss'}{s^2+\Omega^2}\Big)\Big]-2\lambda r f'(g-s)(2g-s)\Big\}_{r=r_2}=0 \qquad (4.6)$$

For $G_1 = 0$ the roots are similar to those that are defined by equation (4.3). We shall investigate now the roots (4.6), which are related to the presence of the outside boundary. After simple transformations we obtain (for $r_0 = r_2$, $g_2 = 1$)



$$q_2 = \sqrt{\frac{\Omega}{s_2^2+\Omega^2}}\Big\{\frac{2(1-s)(2-s)}{\Omega^2+2s^2-2s}-\frac{s(rg'+2)}{\Omega^2}-\frac{1}{2}+\frac{rq'}{2q}-\frac{rss'}{2(s^2+\Omega^2)}\Big\}_{r=r_2} \qquad (4.7)$$

There are real as well as complex roots Ω. For example, if $s^2 \gg 1$, the complex roots are such that $\Omega^2 + 2s^2 - 2s$ is not large (with the exception of the case of small Ω^2, in which instance equation (4.2) is valid and $\mathrm{Re}\,\Omega < 0$). With regard to the limitation of the argument of the root in (2.10), it is easily shown, that to these roots there correspond oscillations ($\mathrm{Re}\,\Omega < 0$), vanishing with time.

Consider now the real positive roots Ω of equation (4.7). We shall establish the functional dependence of s_2 upon $1/q_2$ for a fixed Ω and upon the other parameters of the problem. In studying the influence of the magnitude of a discontinuity of the current density on the boundary, we introduce the assumption quite unessential for the general character of the curves, that the three last terms in the square bracket of (4.7) vanish. For the case of $\Omega = 0.2$ the curves are represented in the figure. The numbers denote values of the parameter $r_2 g_2' + 2$, proportional to the zth component of the current density on the outside boundary layer.

In the case $q = \infty$ the entire region of instability corresponds to the sheet $0 < s_2 < 1$. For the chosen value $\Omega = 0.2$ there are two points $s_2 = 0.98$ and $s_2 = 0.02$. Curves that originate from the first point are practically independent of the magnitude of $r_2 g_2' + 2$ and run approximately parallel to the x-axis. The curves originating at the point $s_2 = 0.02$ are of a different character. For $r_2 g_2' > -2$ they extend into the region of negative s_2 and they have a vertical asymptote for the value q_2, as obtained from formula (4.4); for $r_2g_2' < -2$ the curves extend into the region of positive s_2, and the asymptote again is given by equation (4.4). The curves, however, are discontinuous. For $r_2g_2' = -2$, when the longitudinal current density does not have a discontinuity on the boundary, the distribution of the regions of the instability depends weakly upon the conductivity of the medium. It is to be noted, however, that as the current density is decreased to zero the conductivity also falls abruptly, therefore, the formulas derived above for the case of small magnitudes $r_2g_2' + 2$ may turn out to be inapplicable.

Consequently unstable oscillations, corresponding to equations (4.3) and (4.4), arise from the presence of the discontinuities in the spatial current density distribution. For the type of oscillations chosen the instability may not occur even in the case of an abrupt variation of the current $\mathbf{j}$ on the boundary, if in this region $(\mathbf{i}_r \times \mathbf{j})\cdot\nabla\mathbf{H}_\varphi^* = 0$. For example, for $m = 0$ it is sufficient that dH_z/dr be a continuous function at the boundary.

5. On the stability of a solid string. The method of asymptotic solutions of equations of the magnetohydrodynamics derived above is inapplicable in the case of a solid well-conducting cylinder because of the presence of the singularity at $r = 0$. It is possible get rid of this limitation by a slight variation in the method.

In the region of small r function $H_\varphi(r)$ usually behaves nearly linearly and $H_z(r)$ is approximately constant. Consequently, here a general solution is valid, known [1,2] for the case of q = const, h = const, s = const

$$\mathbf{H}^* = \sum_{p=1}^{3} \{\nabla L_p + iy_p \mathbf{i}_z \times \nabla L_p - iky_p^2 L_p \mathbf{i}_z\} \tag{5.1}$$

where

$$L_p = C_p I_m (kr\sqrt{1-y_p^2}) + D_p K_m (kr\sqrt{1-y_p^2}) \tag{5.2}$$

where C_p and D_p are constants and y_p are the solutions of equation

$$y_p^3 + \frac{q^2(s^2+\Omega^2)}{k^2 r_0^2 \Omega} y_p - \frac{2sq^2}{k^2 r_0^2 \Omega} = 0 \qquad (p = 1, 2, 3) \tag{5.3}$$

For $q \gg 1$ there are two roots $y_p (p = 1, 2)$, for which in the main approximation

$$kr\sqrt{1-y_p^2} = \frac{qr}{r_0}\sqrt{\frac{s^2+\Omega^2}{\Omega}}$$

We may look for two solutions of (2.1) and (2.2) bounded at zero in the form analogous to (2.4), but with substitution of exp λf by Bessel functions chosen so that for small r they will transform into the two solutions (p = 1, 2) indicated above. The conditions for r are fulfilled automatically, therefore, it is sufficient to satisfy the system (2.1) to (2.2) to the corresponding approximation in terms of small parameter λ^{-1}. For example, for the two components $H_\varphi{}^*$ and $H_z{}^*$ two solutions are written in the form

$$H_\varphi{}^* = \lambda F' \sqrt{F}\{(X + \ldots) I_m'(\lambda F) + \lambda^{-1} Y^{(01)} I_m(\lambda F) + \ldots\}$$
$$H_z{}^* = \mp i\lambda F' \sqrt{F}\{(X + \ldots) I_m(\lambda F) + \lambda^{-1} Z^{(01)} I_m'(\lambda F) + \ldots\} \quad (5.4)$$

where

$$\lambda F = \frac{1}{r_0}\int_0^r q\sqrt{\frac{s^2+\Omega^2}{\Omega}}\,dr \pm ik\int_0^r \frac{sg\,dr}{s^2+\Omega^2} + \ldots, \quad X = \frac{C}{\sqrt{rq}}(s^2+\Omega^2)^{-\frac{3}{4}}$$
$$Y^{(01)},\ Z^{(01)} \sim X$$

It is easily shown, that equation (4.4) which defines the instability remains valid in the case of a solid twist.

Note that from the expressions (5.4) there follows that the expansion in the form (2.4) is valid for the condition $|\lambda F(r_1)| \gg 1$.

6. The case of a compressible medium. Let us investigate the stability of a compressible pipe-like string under the assumption, that $p(r)$ and $\rho(r)$ nowhere become zero. All the other starting conditions are the same as in Section 2.

We shall look for the solution of (1.7) to (1.9) in the form of (2.4) to (2.5), assuming that Ω^2 will be a small parameter of the order of λ^{-1}. Equations (2.3), (2.6) and (2.7) remain valid and we obtain for Y and Z the system

$$2Y' + \left[\frac{q'}{q} - \frac{5\Omega'}{2\Omega} + \frac{3(ss'+\Omega\Omega')}{s^2+\Omega^2} + \frac{1}{r}\right]Y + \frac{2mgs}{r(s^2+\Omega^2)}Z +$$
$$+ \frac{mgsp'}{\gamma p(s^2+\Omega^2)}(Y+Z) - \frac{rg\Omega^2 H_0{}^2\lambda f'}{4\pi\gamma p r_0{}^2(s^2+\Omega^2)}\left(rgY + \frac{mr_0 h}{kr}Z\right) = 0 \quad (6.1)$$

$$2Z' + \left[\frac{q'}{q} - \frac{5\Omega'}{2\Omega} + \frac{3(ss'+\Omega\Omega')}{s^2+\Omega^2} - \frac{1}{r}\right]Z - \frac{2k^2 rgs}{m(s^2+\Omega^2)}Y +$$
$$+ \frac{kr_0 hsp'}{\gamma p(s^2+\Omega^2)}(Y+Z) - \frac{krh\Omega^2 H_0{}^2\lambda f'}{4\pi m\gamma p r_0(s^2+\Omega^2)}\left(rgY + \frac{mr_0 h}{kr}Z\right) = 0$$

Expressions (2.10), (3.1) to (3.3) retain their form for the compressible string (it is only necessary that in the equation we substiture for D_{nj} the term $(1/T_j - \chi_j/T_j^{(*)})$ in place of $(1 - \chi_j)/T_j)$. The right-hand side of (3.4) is altered somewhat, but the proportionality between G_j and N_j remains. In the first approximation the coefficients N_j are equal up to a constant to the determinant of the fundamental system of solutions (6.1), taken for $r = r_j$, therefore (3.3) may be divided by $N_1 N_2$. The equation obtained in this manner represents two first terms of the expansion of the exact dispersion relation in terms of the order of λ^{-1}.

In first approximation the roots of the dispersion relation are determined either by equation (4.1), or by the expressions (4.2) where $\Omega_j = \Omega_0 \sqrt{(\rho_j/\rho_0)}$ is substituted in place of Ω. To investigate (4.1) we must find the solutions of (1.7) to (1.9) using the approximation of ideal conductivity. From (4.2) we obtain

$$q_j = (-1)^{j-1} \left\{ \frac{r_0 s\,(rs' + 2mg)}{r\Omega \sqrt{\Omega\,(s^2 + \Omega^2)}} \right\}_{r=r_j} \qquad (j = 1,\ 2) \qquad (6.2)$$

so that the build-up time of the instability motions, arising from the presence of the discontinuities of the spatial current density, does not depend on the assumption of incompressibility of the medium. In the case of the solution of (6.2) the condition of smallness of Ω^2 is satisfied.

BIBLIOGRAPHY

1. Tayler, R.J., Stability of twisted magnetic fields in a fluid of finite electrical conductivity. *Rev. Mod. Phys.* 32, p.907, 1960.
2. Breus, S.I., Ob ustoichivosti zhidkogo tsilindra s tokom pri konechnoi provodimosti (On the stability of a fluid cylinder with a current for finite conductivity). *ZhTF* Vol. 30, p. 1030, 1960.
3. Jukes, J.D., Stability of the sharp pinch and unpinch with finite conductivity. *Phys. Fluids* 4, No. 12, p. 1527, 1961.
4. Vandakurov, Iu.V., K teorii ustoichivosti zhidkogo provodiashchego tsilindra v magnitnom pole (On the theory of stability of a fluid conducting cylinder in a magnetic field). *PMM* Vol. 26, No. 5, 1962.

Translated by J.R.W.